Discrete Symmetry Reduction

Domenico Lippolis

February 8, 2022

Chaos Course 2022

Lorenz system

- Climate model

$$\begin{aligned} \dot{x} &= \sigma\,(y - x) \\ \dot{y} &= \rho\, x - y - xz \\ \dot{z} &= xy - bz \end{aligned} \tag{1}$$

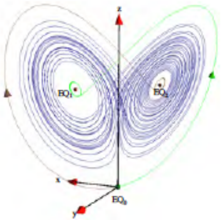
ChaosBook.org

Lorenz system

- Climate model

$$\begin{aligned} \dot{x} &= \sigma\,(y - x) \\ \dot{y} &= \rho\,x - y - xz \\ \dot{z} &= xy - bz \end{aligned} \tag{1}$$

- gives rise to a double-lobed chaotic attractor ($\sigma = 10, b = 8/3, \rho = 28$)



ChaosBook.org

Lorenz system

- As the shape of the attractor suggests, the system is equivariant under the $\mathbf{Z}_2$ symmetry ($\pi-$rotation about the $z-$axis)

$$r(x, y, z) = (-x, -y, z) \tag{2}$$

Lorenz system

- As the shape of the attractor suggests, the system is equivariant under the $\mathbf{Z}_2$ symmetry ($\pi-$rotation about the $z-$axis)

$$r(x, y, z) = (-x, -y, z) \tag{2}$$

- Equivariance means that given a symmetry operation g and the solution to the system $f^t(x)$,

$$f^t\,(g\,\mathbf{x}) = g\,f^t\,(\mathbf{x}) \tag{3}$$

Lorenz system

- As the shape of the attractor suggests, the system is equivariant under the $\mathbf{Z}_2$ symmetry ($\pi-$rotation about the $z-$axis)

$$r(x, y, z) = (-x, -y, z) \tag{2}$$

- Equivariance means that given a symmetry operation g and the solution to the system $f^t(x)$,

$$f^t\,(g\,\mathbf{x}) = g\,f^t\,(\mathbf{x}) \tag{3}$$

- Here $g\,(x, y, z) = r(x, y, z)$, and

$$v(g\,\mathbf{x}) = \begin{pmatrix} \sigma\,(x-y) \\ -\rho\,x + y + xz \\ xy - bz \end{pmatrix} = g\,v(\mathbf{x}) \tag{4}$$

Lorenz system

- As the shape of the attractor suggests, the system is equivariant under the $\mathbf{Z}_2$ symmetry ($\pi-$rotation about the $z-$axis)

$$r(x, y, z) = (-x, -y, z) \tag{2}$$

- Equivariance means that given a symmetry operation g and the solution to the system $f^t(x)$,

$$f^t\,(g\,\mathbf{x}) = g\,f^t\,(\mathbf{x}) \tag{3}$$

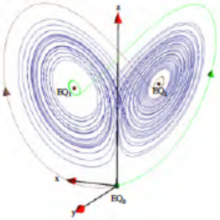
- Here $g\,(x, y, z) = r(x, y, z)$, and

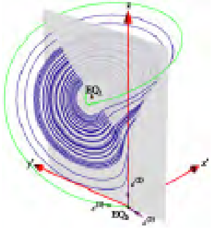
$$v(g\,\mathbf{x}) = \begin{pmatrix} \sigma\,(x - y) \\ -\rho\,x + y + xz \\ xy - bz \end{pmatrix} = g\,v(\mathbf{x}) \tag{4}$$

- Then equivariance follows from $f^t(\mathbf{x}) = \mathbf{x}_0 + \int_0^t d\tau\, v\,[\mathbf{x}(\tau)]$

Lorenz system - symmetry reduction

- Intuitively, the two-lobe attractor would be reduced to one lobe-

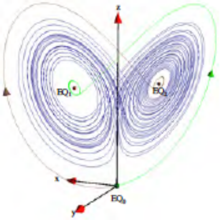


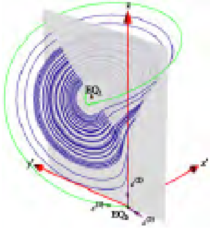


ChaosBook.org

Lorenz system - symmetry reduction

- Intuitively, the two-lobe attractor would be reduced to one lobe-





- given a Poincaré section parallel to the $z-$axis, the fundamental domain $\hat{\mathcal{M}}$ is the half-space between the viewer and the section. Then the full flow is captured by reinjecting back into $\hat{\mathcal{M}}$ every trajectory that exits it, by a $\pi-$rotation about the $z-$axis. But how to realize that?

ChaosBook.org

Lorenz system - symmetry reduction

- Rewrite the Lorenz system in cylindrical coordinates (R, θ, z):

$$\begin{aligned}\dot{R} &= \frac{R}{2}\left[-\sigma - 1 + (\sigma + \rho - z)\sin 2\theta + (1 - \sigma)\cos 2\theta\right] \\ \dot{\theta} &= \frac{1}{2}\left[-\sigma + \rho - z + (\sigma - 1)\sin 2\theta + (\sigma + \rho - z)\cos 2\theta\right] \\ \dot{z} &= -bz + \frac{R^2}{2}\sin 2\theta \end{aligned} \qquad (5)$$

Lorenz system - symmetry reduction

- Rewrite the Lorenz system in cylindrical coordinates (R, θ, z):

$$\begin{aligned} \dot{R} &= \frac{R}{2}\left[-\sigma - 1 + (\sigma + \rho - z)\sin 2\theta + (1-\sigma)\cos 2\theta\right] \\ \dot{\theta} &= \frac{1}{2}\left[-\sigma + \rho - z + (\sigma - 1)\sin 2\theta + (\sigma + \rho - z)\cos 2\theta\right] \\ \dot{z} &= -bz + \frac{R^2}{2}\sin 2\theta \end{aligned} \tag{5}$$

- Notably, everything depends on 2θ and not just θ. We have

$$R = \sqrt{x^2 + y^2} = \sqrt{(-x)^2 + (-y)^2} \tag{6}$$

$$\sin 2\theta = 2\cos\theta\sin\theta = 2\frac{x}{R}\frac{y}{R} = 2\frac{-x}{R}\frac{-y}{R} \tag{7}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \left(\frac{x}{R}\right)^2 - \left(\frac{y}{R}\right)^2 = \left(\frac{-x}{R}\right)^2 - \left(\frac{-y}{R}\right)^2 \tag{8}$$

Lorenz system - symmetry reduction

- we can infer that the coordinates (R, θ, z) quotient the $\mathbf{Z}_2$ symmetry out of the Lorenz flow, that is

$$f^t\left[r(R,\theta,z)\right] = r\left(f^t\left[R,\theta,z\right]\right) = f^t(R,\theta,z) \tag{9}$$

Lorenz system - symmetry reduction

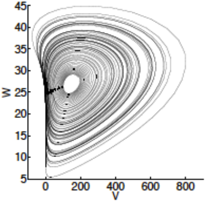
- we can infer that the coordinates (R, θ, z) quotient the $\mathbf{Z}_2$ symmetry out of the Lorenz flow, that is

$$f^t\left[r(R,\theta,z)\right] = r\left(f^t\left[R,\theta,z\right]\right) = f^t(R,\theta,z) \tag{9}$$

- That's no reduction to the fundamental domain, yet. But due to the 2θ dependence we can now guess a set of coordinates

$$(V, W, z) = (R\cos 2\theta, R\sin 2\theta, z) = \left(\frac{x^2-y^2}{R}, \frac{2xy}{R}, z\right) \tag{10}$$

that will do the job (*cf.* exercise 11.5 chaosbook)



Van der Pol oscillator

- One doesn't need chaos to reduce symmetries. Consider the nonlinearly damped oscillator

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0 \tag{11}$$

Van der Pol oscillator

- One doesn't need chaos to reduce symmetries. Consider the nonlinearly damped oscillator

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0 \tag{11}$$

- or equivalently

$$\begin{aligned} \dot{x} &= \mu\left(x - \frac{1}{3}x^3 - y\right) \\ \dot{y} &= \frac{1}{\mu}x \end{aligned} \tag{12}$$

Van der Pol oscillator

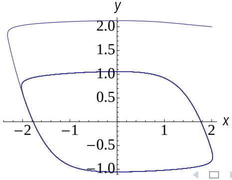
- One doesn't need chaos to reduce symmetries. Consider the nonlinearly damped oscillator

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0 \tag{11}$$

- or equivalently

$$\begin{aligned} \dot{x} &= \mu\left(x - \frac{1}{3}x^3 - y\right) \\ \dot{y} &= \frac{1}{\mu}x \end{aligned} \tag{12}$$

- The dynamics quickly ($\mu = 3$) converges to a limit cycle



Van der Pol oscillator- symmetry reduction

- the Van der Pol oscillator is $\mathbf{Z}_2$-symmetric (or equivariant):

$$v\left[r(x,y,z)\right] = \left[\begin{array}{c} \mu\left(-x+\frac{1}{3}x^3+y\right) \\ \frac{1}{\mu}(-x) \end{array}\right] = -v(x,y,z) \qquad (13)$$

Van der Pol oscillator- symmetry reduction

- the Van der Pol oscillator is $\mathbf{Z}_2$-symmetric (or equivariant):

$$v\left[r(x,y,z)\right] = \left[\begin{array}{c} \mu\left(-x+\frac{1}{3}x^3+y\right) \\ \frac{1}{\mu}(-x) \end{array}\right] = -v(x,y,z) \qquad (13)$$

- in polar coordinates

$$\begin{aligned} \dot{R} &= \frac{1}{R}\left[\mu\, R^2\cos^2\theta - \frac{\mu}{3}R^4\cos^4\theta + \left(\frac{1}{\mu}-\mu\right)R^2\sin 2\theta\right] \\ \dot{\theta} &= \frac{1}{1+\frac{\sin^2\theta}{\cos^2\theta}}\left[\frac{1}{\mu} - \frac{\mu}{\cos^2\theta}\left(\frac{1}{2}\sin 2\theta(1-\frac{1}{3}\cos^2\theta) - \sin^2\theta\right)\right] \qquad (14) \end{aligned}$$

Van der Pol oscillator- symmetry reduction

- the Van der Pol oscillator is $\mathbf{Z}_2$-symmetric (or equivariant):

$$v\left[r(x,y,z)\right] = \begin{bmatrix} \mu\left(-x+\frac{1}{3}x^3+y\right) \\ \frac{1}{\mu}(-x) \end{bmatrix} = -v(x,y,z) \tag{13}$$

- in polar coordinates

$$\begin{aligned} \dot{R} &= \frac{1}{R}\left[\mu\, R^2\cos^2\theta - \frac{\mu}{3}R^4\cos^4\theta + \left(\frac{1}{\mu}-\mu\right)R^2\sin 2\theta\right] \\ \dot{\theta} &= \frac{1}{1+\frac{\sin^2\theta}{\cos^2\theta}}\left[\frac{1}{\mu} - \frac{\mu}{\cos^2\theta}\left(\frac{1}{2}\sin 2\theta(1-\frac{1}{3}\cos^2\theta) - \sin^2\theta\right)\right] \end{aligned} \tag{14}$$

- Again, that depends on 2θ and we can make the same change of coordinates as for the Lorenz system.

  Exercise: plot the symmetry-reduced Van der Pol limit cycle

Transfer operator

- Alternatively to ODEs, one can follow as swarm of trajectories, obeying the Liouville equation

$$\partial_t \rho(x,t) + \nabla \cdot [\rho(x,t)\, v(x)] = 0 \tag{15}$$

ChaosBook.org

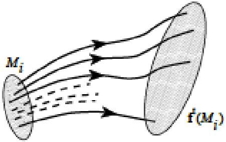
Transfer operator

- Alternatively to ODEs, one can follow as swarm of trajectories, obeying the Liouville equation

$$\partial_t \rho(x,t) + \nabla \cdot [\rho(x,t)\, v(x)] = 0 \qquad (15)$$

- or, equivalently, $\rho(x,t)$ is transported by the flow $f^t(x)$ via the Perron-Frobenius operator

$$\rho(x,t) = \left(\mathcal{L}^t \circ \rho\right)(x) = \int dx_0\, \delta\left(x - f^t(x_0)\right) \rho(x_0, 0) \qquad (16)$$



ChaosBook.org

Transfer operator

- Symmetry (commutation) condition

$$gf(x) = f(gx)\,, \tag{17}$$

Transfer operator

- Symmetry (commutation) condition

$$gf(x) = f(gx)\,, \tag{17}$$

- applied to the transfer operator

$$\mathcal{L}(y,x) = \int dx\,\delta\,(y - f(x))\circ \tag{18}$$

Transfer operator

- Symmetry (commutation) condition

$$gf(x) = f(gx)\,, \tag{17}$$

- applied to the transfer operator

$$\mathcal{L}(y,x) = \int dx\,\delta\,(y - f(x))\circ \tag{18}$$

- constraining densities on the orbit $y = f(x)$ so that for a $g-$symmetric flow

$$\mathcal{L}(gy,gx) = \int dgx\,\delta\,(gy - f(gx))\circ = \int dgx\,\delta\,(g\,[y - f(x)])\circ \tag{19}$$

still constrains the dynamics on $y = f(x)$.

Transfer operator

- Moreover

$$\begin{aligned}
\mathcal{L}(gy, gx)\,\rho(x) &= \int dgx\,\delta\left(gy - f(gx)\right)\,\rho(x) \\
&= \int_{g(\mathbb{R})} du\,\delta\left(gy - f(u)\right)\,\rho(f^{-1}(u)) \\
&= \sum \rho(g^{-1}f^{-1}(gy))\left[g^{-1}f^{-1}(gy)\right]' \\
&= \sum_{y=f^{-1}(x)} \frac{\rho(f^{-1}(y))}{|f'(y)|} = \mathcal{L}(y, x)\,\rho(x) \qquad (20)
\end{aligned}$$

Transfer operator

- Moreover

$$\begin{aligned}\mathcal{L}(gy, gx)\,\rho(x) &= \int dgx\,\delta\left(gy - f(gx)\right)\,\rho(x) \\ &= \int_{g(\mathbb{R})} du\,\delta\left(gy - f(u)\right)\,\rho(f^{-1}(u)) \\ &= \sum \rho(g^{-1}f^{-1}(gy))\left[g^{-1}f^{-1}(gy)\right]' \\ &= \sum_{y=f^{-1}(x)} \frac{\rho(f^{-1}(y))}{|f'(y)|} = \mathcal{L}(y,x)\,\rho(x) \qquad (20)\end{aligned}$$

- Thus the symmetry condition $f(x) = g^{-1}f(g\,x)$ is equivalent to $\mathcal{L}(gy, gx) = \mathcal{L}(y, x)$

Example

- Consider a $\mathbf{Z}_2$ symmetry operation C for a 2D map $x_{n+1} = f(x_n)$, such that

$$f(Cx) = Cf(x), \qquad C = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{21}$$

Example

- Consider a $\mathbf{Z}_2$ symmetry operation C for a 2D map $x_{n+1} = f(x_n)$, such that

$$f(Cx) = Cf(x), \qquad C = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{21}$$

- The phase space can be decomposed into symmetric and antisymmetric spaces by means of the projectors

$$P_{A_1} = \frac{1}{2}(e + C), \qquad P_{A_2} = \frac{1}{2}(e - C) \tag{22}$$

Example

- Consider a $\mathbf{Z}_2$ symmetry operation C for a 2D map $x_{n+1} = f(x_n)$, such that

$$f(Cx) = Cf(x)\,, \qquad C = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \tag{21}$$

- The phase space can be decomposed into symmetric and antisymmetric spaces by means of the projectors

$$P_{A_1} = \frac{1}{2}(e + C)\,, \qquad P_{A_2} = \frac{1}{2}(e - C) \tag{22}$$

- Applied to the transfer operator to obtain

$$\mathcal{L}_{A_1} = P_{A_1}\mathcal{L}(y,x) = \frac{1}{2}\left[\mathcal{L}(y,x) + \mathcal{L}(-y,x)\right] \tag{23}$$

$$\mathcal{L}_{A_2} = P_{A_2}\mathcal{L}(y,x) = \frac{1}{2}\left[\mathcal{L}(y,x) - \mathcal{L}(-y,x)\right] \tag{24}$$

Transfer operator decomposition

- In general, consider the projector

$$P_\alpha = \frac{d_\alpha}{|G|} \sum_h \chi_\alpha(h) h^{-1} \tag{25}$$

P Cvitanović and B Eckhardt, Nonlinearity **6**, 277 (1993)

Transfer operator decomposition

- In general, consider the projector

$$P_\alpha = \frac{d_\alpha}{|G|} \sum_h \chi_\alpha(h) h^{-1} \tag{25}$$

- with group characters

$$\chi_\alpha(h) = \mathrm{tr}\, D_\alpha(h) \tag{26}$$

P Cvitanović and B Eckhardt, Nonlinearity **6**, 277 (1993)

Transfer operator decomposition

- In general, consider the projector

$$P_\alpha = \frac{d_\alpha}{|G|} \sum_h \chi_\alpha(h) h^{-1} \tag{25}$$

- with group characters

$$\chi_\alpha(h) = \mathrm{tr}\, D_\alpha(h) \tag{26}$$

- The P_α's split the transfer operator into a sum of irreducible space contributions, each

$$\mathcal{L}_{A_\alpha}(y, x) = \frac{d_\alpha}{|G|} \sum_{h \in G} \chi_\alpha(h) \mathcal{L}(h^{-1} y, x) \tag{27}$$

P Cvitanović and B Eckhardt, Nonlinearity **6**, 277 (1993)

Transfer operator decomposition

- In the example, the $\mathbf{Z}_2$ group has two irreps, such that

$$\begin{aligned} \chi_1(e) &= 1\,, \;\; \chi_1(C) = 1 \\ \chi_2(e) &= 1\,, \;\; \chi_2(C) = -1 \end{aligned}$$

which yields

$$P_{A_1} = \frac{1}{2}(e + C)\,, \qquad P_{A_2} = \frac{1}{2}(e - C) \tag{28}$$

Transfer operator decomposition

- In the example, the $\mathbf{Z}_2$ group has two irreps, such that

$$\begin{aligned} \chi_1(e) &= 1\,,\quad \chi_1(C) = 1 \\ \chi_2(e) &= 1\,,\quad \chi_2(C) = -1 \end{aligned}$$

which yields

$$P_{A_1} = \frac{1}{2}(e + C)\,, \qquad P_{A_2} = \frac{1}{2}(e - C) \tag{28}$$

- Eigenvalues of the transfer operator are evaluated by means of traces, also in the fundamental domain as

$$\mathrm{tr}\,\mathcal{L} = \int_M dx\,\mathcal{L}(x, x) = \int_{\tilde{M}} d\tilde{x} \sum_h \mathrm{tr}\,D(h)\,\mathcal{L}(h^{-1}\tilde{x}, \tilde{x}) \tag{29}$$

Transfer operator decomposition

- In the example, the $\mathbf{Z}_2$ group has two irreps, such that

$$\begin{aligned} \chi_1(e) &= 1\,, \quad \chi_1(C) = 1 \\ \chi_2(e) &= 1\,, \quad \chi_2(C) = -1 \end{aligned}$$

which yields

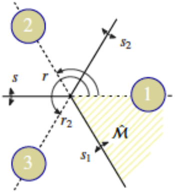
$$P_{A_1} = \frac{1}{2}(e + C)\,, \qquad P_{A_2} = \frac{1}{2}(e - C) \tag{28}$$

- Eigenvalues of the transfer operator are evaluated by means of traces, also in the fundamental domain as

$$\mathrm{tr}\,\mathcal{L} = \int_M dx\,\mathcal{L}(x,x) = \int_{\tilde{M}} d\tilde{x} \sum_h \mathrm{tr}\,D(h)\,\mathcal{L}(h^{-1}\tilde{x},\tilde{x}) \tag{29}$$

- Contributions to the trace come from periodic orbits

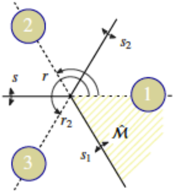
Three-disk scatterer



- Symmetry group is D_3, dihedral

$$D_3 = \{e, r, r_2, s, s_1, s_2\} \quad (30)$$

Three-disk scatterer



- Symmetry group is D_3, dihedral

$$D_3 = \{e, r, r_2, s, s_1, s_2\} \tag{30}$$

- in matrix representation

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad r = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{pmatrix}, \quad r_2 = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}$$

$$s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad s_1 = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}, \quad s_2 = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$$

D_3 group properties

- D_3 is *not* Abelian, for example

$$s \circ s_1 = r \neq s_1 \circ s = r_2 \tag{31}$$

ChaosBook.org

D_3 group properties

- D_3 is *not* Abelian, for example

$$s \circ s_1 = r \neq s_1 \circ s = r_2 \tag{31}$$

- mulitiplication table

D_3	1	r	r_2	s	s_1	s_2
1	1	r	r_2	s	s_1	s_2
r	r	r_2	1	s_2	s	s_1
r_2	r_2	1	r	s_1	s_2	s
s	s	s_1	s_2	1	r	r_2
s_1	s_1	s_2	s	r_2	1	r
s_2	s_2	s	s_1	r	r_2	1

ChaosBook.org

D_3 group properties

- D_3 is *not* Abelian, for example

$$s \circ s_1 = r \neq s_1 \circ s = r_2 \tag{31}$$

- mulitiplication table

D_3	1	r	r_2	s	s_1	s_2
1	1	r	r_2	s	s_1	s_2
r	r	r_2	1	s_2	s	s_1
r_2	r_2	1	r	s_1	s_2	s
s	s	s_1	s_2	1	r	r_2
s_1	s_1	s_2	s	r_2	1	r
s_2	s_2	s	s_1	r	r_2	1

- D_3 has six subgroups...

$$\{e\}\,,\ \{e,s\}\,,\ \{e,s_1\}\,,\ \{e,s_2\}\,,\ \{e,r,r_2\} \tag{32}$$

ChaosBook.org

D_3 group properties

- D_3 is *not* Abelian, for example

$$s \circ s_1 = r \neq s_1 \circ s = r_2 \tag{31}$$

- mulitiplication table

D_3	1	r	r_2	s	s_1	s_2
1	1	r	r_2	s	s_1	s_2
r	r	r_2	1	s_2	s	s_1
r_2	r_2	1	r	s_1	s_2	s
s	s	s_1	s_2	1	r	r_2
s_1	s_1	s_2	s	r_2	1	r
s_2	s_2	s	s_1	r	r_2	1

- D_3 has six subgroups...

$$\{e\}\,,\ \{e,s\}\,,\ \{e,s_1\}\,,\ \{e,s_2\}\,,\ \{e,r,r_2\} \tag{32}$$

- ...And three classes

$$\{e\}\,,\ \{s,s_1,s_2\}\,,\ \{r,r_2\} \tag{33}$$

ChaosBook.org

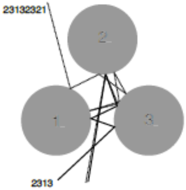
Cycles and their symmetries

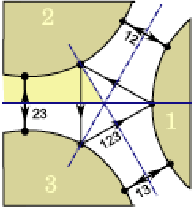
- The pinball is an *open* system, so everything escapes eventually. The transient chaotic dynamics is nailed by the trapped orbits.

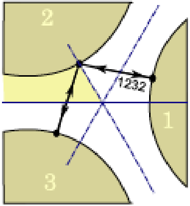
ChaosBook.org

Cycles and their symmetries

- The pinball is an *open* system, so everything escapes eventually. The transient chaotic dynamics is nailed by the trapped orbits.
- It is convenient to describe orbits by symbolic sequences



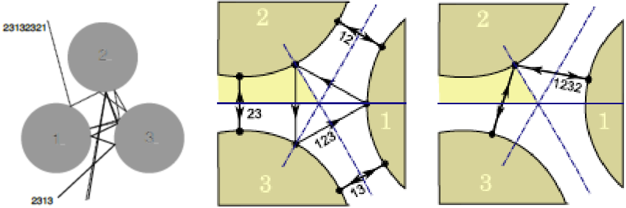




ChaosBook.org

Cycles and their symmetries

- The pinball is an *open* system, so everything escapes eventually. The transient chaotic dynamics is nailed by the trapped orbits.
- It is convenient to describe orbits by symbolic sequences



- Then symmetry operations can be applied directly to symbols, e.g.

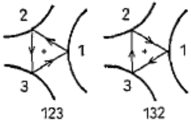
$$r(\overline{12}) = \overline{23}\,,\ r(\overline{23}) = \overline{31}\,,\ r(\overline{31}) = \overline{12} \tag{34}$$

ChaosBook.org

Cycles and their symmetries

- More examples:

$$s_1(\overline{123}) = \overline{132} \tag{35}$$

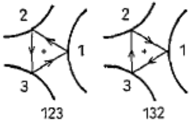


but also: $\overline{123}$ and $\overline{132}$ form a group of invariant cycles under C_3 (rotations)

Cycles and their symmetries

- More examples:

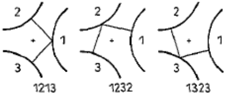
$$s_1(\overline{123}) = \overline{132} \tag{35}$$



but also: $\overline{123}$ and $\overline{132}$ form a group of invariant cycles under C_3 (rotations)

- cycles $\overline{1213}$, $\overline{1232}$ and $\overline{1323}$ are invariant under D_1 (flips), e.g.

$$s_1(\overline{1213}) = \overline{1312} = \overline{1213} \tag{36}$$

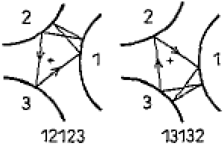


but also

$$r_2(\overline{1213}) = \overline{1232}\,,\;\; r_2(\overline{1232}) = \overline{1323} \tag{37}$$

Cycles and their symmetries

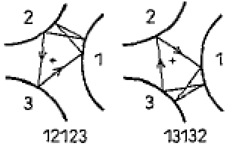
- six-degenerate five-cycle $\overline{12123}$:



$$s(\overline{12123}) = \overline{13132} \tag{38}$$

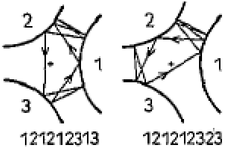
Cycles and their symmetries

- six-degenerate five-cycle $\overline{12123}$:



$$s(\overline{12123}) = \overline{13132} \quad (38)$$

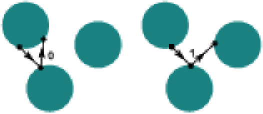
- Besides D_3-symmetries, pinball has time-reversal- :



$$T(\overline{121212313}) = \overline{121212323} \quad (39)$$

Fundamental domain

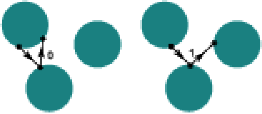
- Reduce three symbols 123 to two: 0 (backward) and 1 (forward)



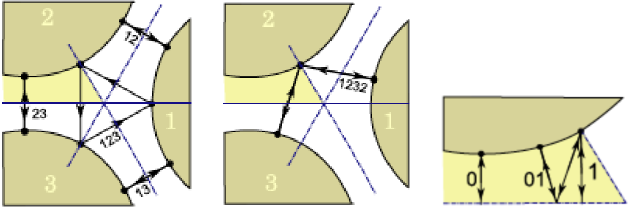
ChaosBook.org

Fundamental domain

- Reduce three symbols 123 to two: 0 (backward) and 1 (forward)



- Restrict the dynamics to one disk + reflection off the symmetry axes:



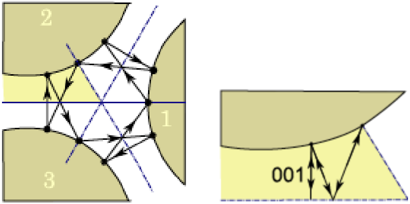
$$\overline{23} \to \overline{0}\,,\ \overline{123} \to \overline{1}\,,\ \overline{1232} \to \overline{01} \tag{40}$$

ChaosBook.org

Fundamental domain

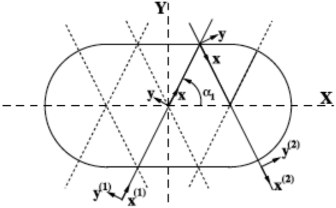
- Time-reversal nine-cycle

$$\overline{121212323} \equiv \overline{121212313} \to \overline{001} \tag{41}$$



ChaosBook.org

Bunimovich stadium

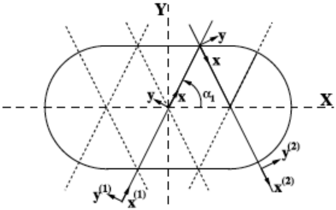


- C_{2v} symmetry group

$$C_{2v} = \{e, s_x, s_y, C_2\}$$

E G Vergini and G G Carlo, J. Phys. A: Math. Gen. **33**, 4717 (2000)

Bunimovich stadium



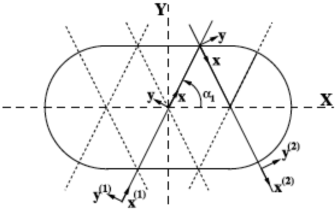
- C_{2v} symmetry group

$$C_{2v} = \{e, s_x, s_y, C_2\}$$

- s_x, s_y flips around $x-$, $y-$axes

E G Vergini and G G Carlo, J. Phys. A: Math. Gen. **33**, 4717 (2000)

Bunimovich stadium



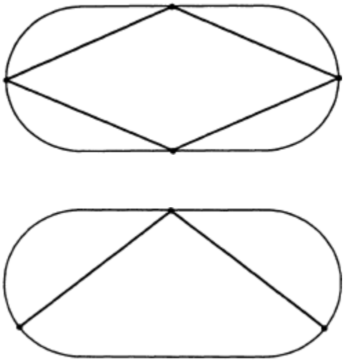
- C_{2v} symmetry group

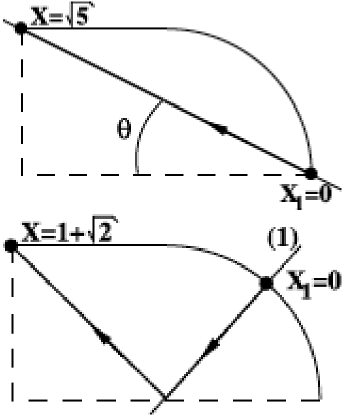
$$C_{2v} = \{e, s_x, s_y, C_2\}$$

- s_x, s_y flips around $x-$, $y-$axes
- C_2 rotation by π

E G Vergini and G G Carlo, J. Phys. A: Math. Gen. **33**, 4717 (2000)

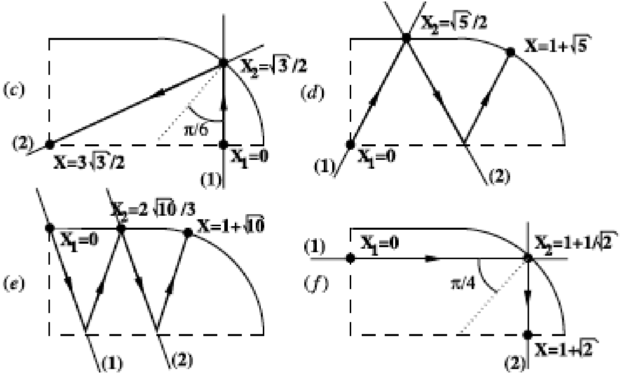
Desymmetrization





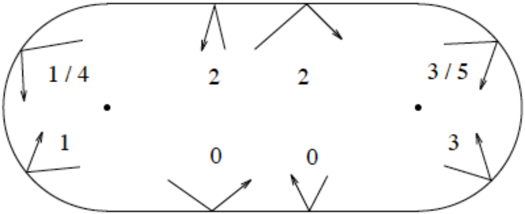
E G Vergini and G G Carlo, J. Phys. A: Math. Gen. **33**, 4717 (2000)

Desymmetrization

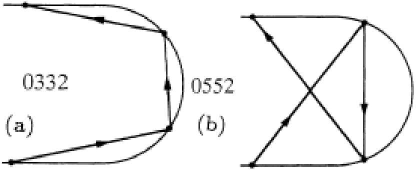


E G Vergini and G G Carlo, J. Phys. A: Math. Gen. **33**, 4717 (2000)

Symbolic dynamics

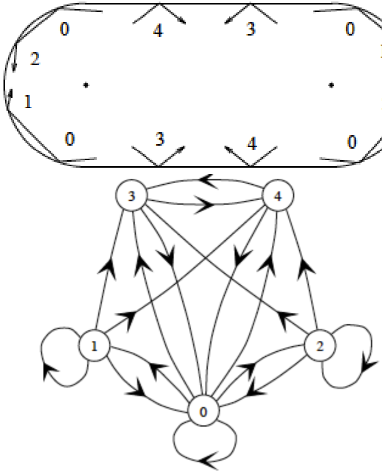


- six symbols to avoid directional ambiguity



K T Hansen and P Cvitanović, chao-dyn/9502005 (1995); O. Biham and M Kvale, Phys. Rev. A **46**, 6334 (1992)

Five-symbol dynamics

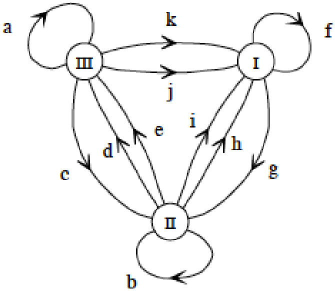


K T Hansen and P Cvitanović, chao-dyn/9502005 (1995)

Symmetry-reduced symbolic dynamics

s_t	$\sigma_{t-1}\sigma_t$	$\sigma_{t-n-1}0^n\sigma_t$
a	11	
	22	
b	00	
c	10	
	20	
d	01	$\_10^n1\_$
	01	$\_40^n1\_$
	02	$\_20^n2\_$
	02	$\_30^n2\_$
e	01	$\_20^n1\_$
	01	$\_30^n1\_$
	02	$\_10^n2\_$
	02	$\_40^n2\_$
f	34	
	43	

s_t	$\sigma_{t-1}\sigma_t$	$\sigma_{t-n-1}0^n\sigma_t$
g	30	
	40	
h	03	$\_20^n3\_$
	03	$\_30^n3\_$
	04	$\_10^n4\_$
	04	$\_40^n4\_$
i	03	$\_10^n3\_$
	03	$\_40^n3\_$
	04	$\_20^n4\_$
	04	$\_30^n4\_$
j	23	
	14	
k	24	
	13	



Coupled Map Lattices

- Dynamical systems with discrete space and time and continuous state variables

$$\Phi_{n+1}^{(i)} = (1-a)f\left(\Phi_n^{(i)}\right) + \frac{a}{2}\left[g\left(\Phi_n^{(i+1)}\right) + g\left(\Phi_n^{(i-1)}\right)\right] \tag{42}$$

Coupled Map Lattices

- Dynamical systems with discrete space and time and continuous state variables

$$\Phi_{n+1}^{(i)} = (1-a) f\left(\Phi_n^{(i)}\right) + \frac{a}{2}\left[g\left(\Phi_n^{(i+1)}\right) + g\left(\Phi_n^{(i-1)}\right)\right] \quad (42)$$

- It becomes a system of N coupled maps, e.g. ($N = 2$, set $f = g$)

$$x_{n+1} = (1-a) f(x_n) + a f(y_n) \quad (43)$$

$$y_{n+1} = (1-a) f(y_n) + a f(x_n) \quad (44)$$

Coupled Map Lattices

- Dynamical systems with discrete space and time and continuous state variables

$$\Phi_{n+1}^{(i)} = (1-a) f\left(\Phi_n^{(i)}\right) + \frac{a}{2}\left[g\left(\Phi_n^{(i+1)}\right) + g\left(\Phi_n^{(i-1)}\right)\right] \tag{42}$$

- It becomes a system of N coupled maps, e.g. ($N = 2$, set $f = g$)

$$x_{n+1} = (1-a) f(x_n) + a f(y_n) \tag{43}$$

$$y_{n+1} = (1-a) f(y_n) + a f(x_n) \tag{44}$$

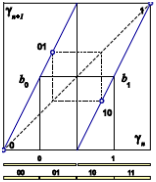
- $a = 0$ yields uncoupled dynamics, that is every lattice site follows its own independent dynamics given by f

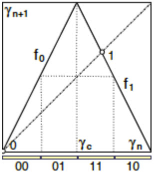
$$x_{n+1} = f(x_n) \tag{45}$$

$$y_{n+1} = f(y_n) \tag{46}$$

Coupled Map Lattices

- Suppose f is chaotic, e.g. Bernoulli shifts ($x_{n+1} = 2x_n \bmod 1$) or conjugated

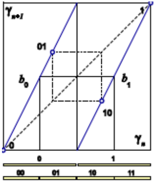


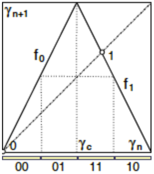


ChaosBook.org

Coupled Map Lattices

- Suppose f is chaotic, e.g. Bernoulli shifts $(x_{n+1} = 2x_n \bmod 1)$ or conjugated





- Take for example the periodic points of period $n_p = 2$: the single map has

$$\begin{matrix} 0 \\ 0 \end{matrix}, \quad \begin{matrix} 0 \\ 1 \end{matrix}, \quad \begin{matrix} 1 \\ 0 \end{matrix}, \quad \begin{matrix} 1 \\ 1 \end{matrix} \tag{47}$$

ChaosBook.org

Coupled Map Lattices

Then the two-dimensional map has $4^2 = 16$ periodic points classified as

- six prime orbits

$$\begin{matrix} 00 \\ 01 \end{matrix}\,, \quad \begin{matrix} 00 \\ 11 \end{matrix}\,, \quad \begin{matrix} 10 \\ 11 \end{matrix}\,, \quad \begin{matrix} 01 \\ 10 \end{matrix}\,, \quad \begin{matrix} 00 \\ 10 \end{matrix}\,, \quad \begin{matrix} 01 \\ 11 \end{matrix} \tag{48}$$

Coupled Map Lattices

Then the two-dimensional map has $4^2 = 16$ periodic points classified as

- six prime orbits

$$\begin{matrix} 00 \\ 01 \end{matrix}, \quad \begin{matrix} 00 \\ 11 \end{matrix}, \quad \begin{matrix} 10 \\ 11 \end{matrix}, \quad \begin{matrix} 01 \\ 10 \end{matrix}, \quad \begin{matrix} 00 \\ 10 \end{matrix}, \quad \begin{matrix} 01 \\ 11 \end{matrix} \tag{48}$$

- their cyclic permutations

$$\begin{matrix} 01 \\ 00 \end{matrix}, \quad \begin{matrix} 11 \\ 00 \end{matrix}, \quad \begin{matrix} 11 \\ 10 \end{matrix}, \quad \begin{matrix} 10 \\ 01 \end{matrix}, \quad \begin{matrix} 10 \\ 00 \end{matrix}, \quad \begin{matrix} 11 \\ 01 \end{matrix} \tag{49}$$

Coupled Map Lattices

Then the two-dimensional map has $4^2 = 16$ periodic points classified as

- six prime orbits

$$\begin{matrix} 00 \\ 01 \end{matrix}, \begin{matrix} 00 \\ 11 \end{matrix}, \begin{matrix} 10 \\ 11 \end{matrix}, \begin{matrix} 01 \\ 10 \end{matrix}, \begin{matrix} 00 \\ 10 \end{matrix}, \begin{matrix} 01 \\ 11 \end{matrix} \tag{48}$$

- their cyclic permutations

$$\begin{matrix} 01 \\ 00 \end{matrix}, \begin{matrix} 11 \\ 00 \end{matrix}, \begin{matrix} 11 \\ 10 \end{matrix}, \begin{matrix} 10 \\ 01 \end{matrix}, \begin{matrix} 10 \\ 00 \end{matrix}, \begin{matrix} 11 \\ 01 \end{matrix} \tag{49}$$

- and four period-one repeated orbits

$$\begin{matrix} 00 \\ 00 \end{matrix}, \begin{matrix} 01 \\ 01 \end{matrix}, \begin{matrix} 10 \\ 10 \end{matrix}, \begin{matrix} 11 \\ 11 \end{matrix} \tag{50}$$

Coupled Map Lattices - symmetry reduction

- A symmetrically coupled lattice of N spatial sites has symmetry group D_N of order $2N$ (spatial cyclic permutation + reflections)

Coupled Map Lattices - symmetry reduction

- A symmetrically coupled lattice of N spatial sites has symmetry group D_N of order $2N$ (spatial cyclic permutation + reflections)
- If $N = 2$, cyclic shift is the same as reflection and $D_2 = C_2 = \{e, s\}$. The prime orbits of period $n_p = 2$ in the fundamental domain are classified as

Coupled Map Lattices - symmetry reduction

- A symmetrically coupled lattice of N spatial sites has symmetry group D_N of order $2N$ (spatial cyclic permutation + reflections)
- If $N = 2$, cyclic shift is the same as reflection and $D_2 = C_2 = \{e, s\}$. The prime orbits of period $n_p = 2$ in the fundamental domain are classified as
- two asymmetric orbits of period two in the original dynamics

$$\begin{matrix} 00 \\ 10 \end{matrix} , \quad \begin{matrix} 10 \\ 11 \end{matrix} \tag{51}$$

Coupled Map Lattices - symmetry reduction

- A symmetrically coupled lattice of N spatial sites has symmetry group D_N of order $2N$ (spatial cyclic permutation + reflections)
- If $N = 2$, cyclic shift is the same as reflection and $D_2 = C_2 = \{e, s\}$. The prime orbits of period $n_p = 2$ in the fundamental domain are classified as
- two asymmetric orbits of period two in the original dynamics

$$\begin{matrix} 00 \\ 10 \end{matrix}, \quad \begin{matrix} 10 \\ 11 \end{matrix} \tag{51}$$

- one boundary orbit $(x = y)$ originally of period two

$$\begin{matrix} 00 \\ 11 \end{matrix} \tag{52}$$

Coupled Map Lattices - symmetry reduction

- and three symmetric orbits of original period four

$$\begin{matrix} 00 \\ 01 \\ 00 \\ 10 \end{matrix}, \quad \begin{matrix} 01 \\ 01 \\ 10 \\ 10 \end{matrix}, \quad \begin{matrix} 10 \\ 11 \\ 01 \\ 11 \end{matrix} \tag{53}$$

Coupled Map Lattices - symmetry reduction

- and three symmetric orbits of original period four

$$\begin{matrix} 00 \\ 01 \\ 00 \\ 10 \end{matrix} \, , \quad \begin{matrix} 01 \\ 01 \\ 10 \\ 10 \end{matrix} \, , \quad \begin{matrix} 10 \\ 11 \\ 01 \\ 11 \end{matrix} \tag{53}$$

- Applying the flip $s : (x, y) \to (y, x)$, the orbits of the last group are all copies of the orbits of the first group, or repeated orbits, and we are left with

$$\begin{matrix} 00 \\ 01 \end{matrix} \, , \quad \begin{matrix} 01 \\ 11 \end{matrix} \, , \quad \begin{matrix} 00 \\ 11 \end{matrix} \tag{54}$$

as prime cycles in the fundamental domain $x \geq y$

Coupled Map Lattices - symmetry reduction

- and three symmetric orbits of original period four

$$\begin{matrix} 00 \\ 01 \\ 00 \\ 10 \end{matrix}, \quad \begin{matrix} 01 \\ 01 \\ 10 \\ 10 \end{matrix}, \quad \begin{matrix} 10 \\ 11 \\ 01 \\ 11 \end{matrix} \tag{53}$$

- Applying the flip $s : (x, y) \to (y, x)$, the orbits of the last group are all copies of the orbits of the first group, or repeated orbits, and we are left with

$$\begin{matrix} 00 \\ 01 \end{matrix}, \quad \begin{matrix} 01 \\ 11 \end{matrix}, \quad \begin{matrix} 00 \\ 11 \end{matrix} \tag{54}$$

as prime cycles in the fundamental domain $x \geq y$

- Exercise: desymmetrize period-two orbits for lattices with $N = 3$ (hence for D_3 symmetry group)