Stretch & Fold

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Chaos Course 2022

Markov partitions

- Given a map f, and a phase space $\mathcal{M}$, we can divide it into finitely many $\mathcal{M}_0, \mathcal{M}_1, ..., \mathcal{M}_{N-1}$, and record $f^n(x)$, $x \in \mathcal{M}$ maps where.

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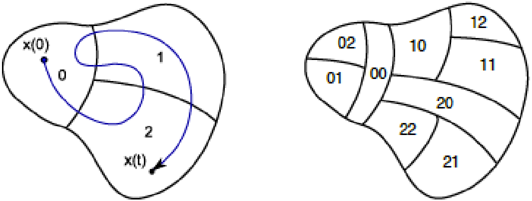
- Difficulties can be
  1. if $\mathcal{M}_i \cap \mathcal{M}_j$ then one point can be coded by multiple sequences
  2. all points in the intersection $\bigcap_{n\in\mathbb{Z}} f^n(\mathcal{M}_{s_n})$ are coded by the same $s = \{s_n\}_{n\in\mathbb{Z}}$



Katok & Hasselblatt, ChaosBook.org

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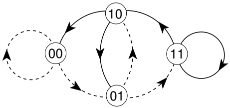
- Example:



σ_A moves the origin of a sequence on A by the next vertex

Markov partitions

If

- Every intersection $\bigcap_{n\in\mathbb{Z}} f^n(\mathcal{M}_{s_n})$ contains no more than one point, one can define

$$h : \Lambda \subset S_N \longrightarrow \mathcal{M} \quad (3)$$

such that

$$f \circ h = h \circ \sigma_N \quad (4)$$

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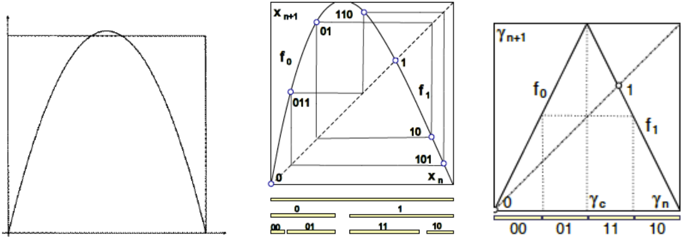
- that is the map f is a factor of some symbolic system
- The decomposition $\mathcal{M}_0, \mathcal{M}_1, ..., \mathcal{M}_{N-1}$ that makes f semiconjugate to σ_A is called a Markov partition

Logistic map: partition

- Consider the quadratic map

$$f(x) = \lambda x(1-x) \quad \lambda > 4 \tag{5}$$



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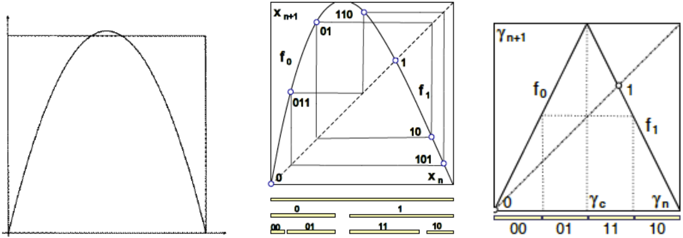

- Here the collection Λ of points with bounded orbits is

$$\Lambda = \bigcap_{n \in \mathbb{Z}} f^{-n}(\mathcal{M}_{s_n}) = \bigcap_{n \in \mathbb{Z}} f^{-n}([0,1])$$

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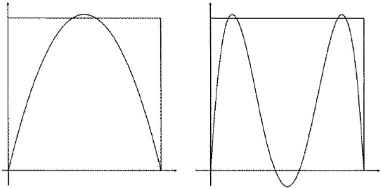

- Then $f^{-1}([0,1]) = \mathcal{M}_0 \cup \mathcal{M}_1$, where

$$\mathcal{M}_0 = \left[0, \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{1}{\lambda}}\right], \quad \mathcal{M}_1 = \left[\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{1}{\lambda}}, 1\right] \qquad (6)$$

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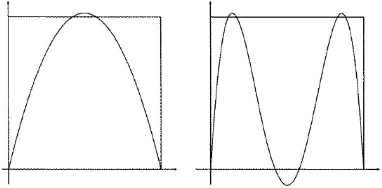

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- and $f^{-2}([0,1]) = \mathcal{M}_{00} \cup \mathcal{M}_{01} \cup \mathcal{M}_{11} \cup \mathcal{M}_{10}$

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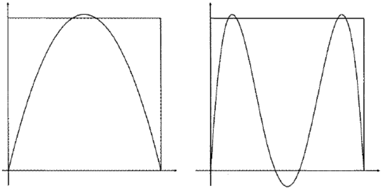

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- and $f^{-2}([0,1]) = \mathcal{M}_{00} \cup \mathcal{M}_{01} \cup \mathcal{M}_{11} \cup \mathcal{M}_{10}$
- Because $|f'(x)| > 1$, everywhere on $\mathcal{M}$, for every sequence s, the diameter of the intersections $\bigcap_{n \in \mathbb{Z}} f^{-n}(\mathcal{M}_{s_n})$ shrinks exponentially

Logistic map: partition

- Then $\Lambda = \bigcap_{n \in \mathbb{Z}} f^{-n}([0,1])$ is a Cantor set for the sequence s, and the intersection

$$h(\{s\}) = \bigcap_{n \in \mathbb{Z}} f^{-n}(\mathcal{M}_{s_n}) \tag{7}$$

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- We can furthermore show that h is a continuous bijection and thus a homeomorphism:

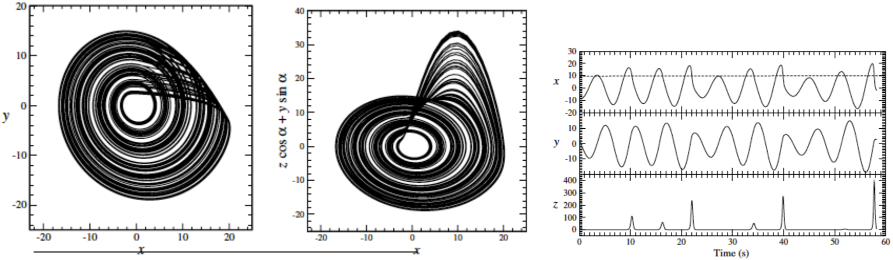
$$h:\ S \longrightarrow \Lambda \tag{8}$$

and thus the symbolic dynamics is isomorphic to the dynamics

Stretch & Fold

- But where does the chaos come from? Here's a prototype example due to Rössler

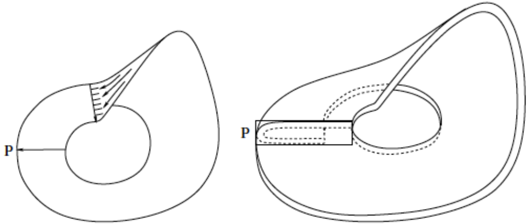
$$
\begin{aligned}
\dot{x} &= -y - z \\
\dot{y} &= x + ay \\
\dot{z} &= b + z(x - c)
\end{aligned}
$$



Rössler & Letellier 2020

Stretch & Fold

- Can understand the dynamics as 'paper flow' or 'cake flow'



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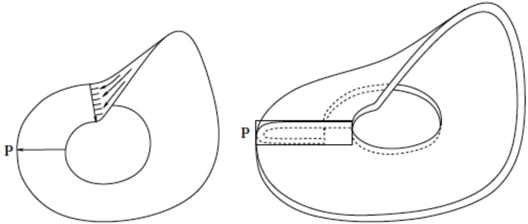
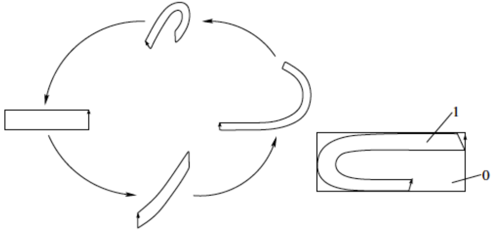
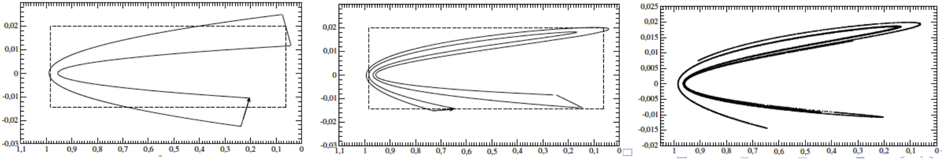


- A $2D$ cross section is rotated, stretched, and folded by the dynamics



Stretch & Fold: return map

- Model a return map $[x_{n+1}, y_{n+1}] = [f_1(x_n, y_n), f_2(x_n, y_n)]$ such that
  1. make x_{n+1} a folded function of x_n
  2. make y_{n+1} non-inverted w.r.t. y_n along the ascending part of $f_1(x_n, y_n)$ and inverted along its descending part



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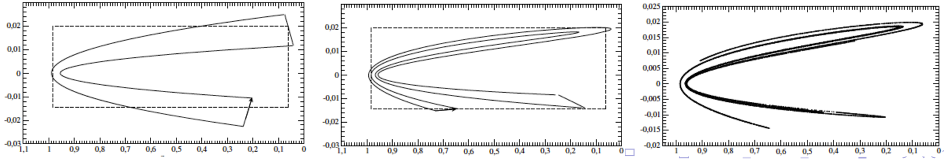
- That translates to, for example,

$$
\begin{aligned}
x_{n+1} &= \lambda\, x_n\,(1 - x_n) - \epsilon\, y_n \\
y_{n+1} &= (\delta\, y_n - \epsilon)\,(1 - 2x_n)
\end{aligned}
\tag{9}
$$



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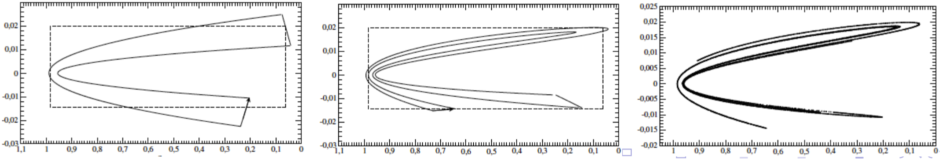
- example: $\lambda = 3.9, \delta = 0.4, \epsilon = 0.02$

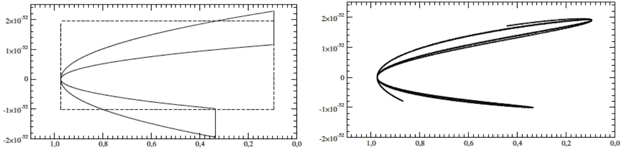


Special cases

1. take $\epsilon = 0$, get

$$x_{n+1} = \lambda\, x_n\, (1 - x_n) \tag{10}$$

- $\epsilon = 0.02 \cdot 10^{-50}$



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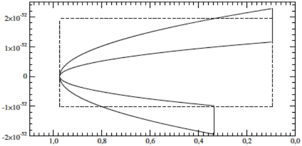

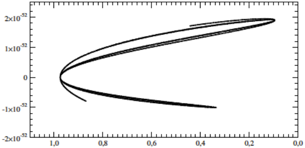

2. $\delta = 0$:

$$\begin{aligned} x_{n+1} &= \lambda\, x_n\, (1 - x_n) - y_n \\ y_{n+1} &= -\epsilon\, (1 - 2x_n) \end{aligned} \tag{11}$$

The Cremona transformation

- still for $\delta = 0$, rescale the variables (for some k) as

$$\begin{aligned} x' &= \frac{1}{k}\left(x - \frac{1}{2}\right) \\ y' &= -\frac{1}{k}\,y \end{aligned} \tag{12}$$

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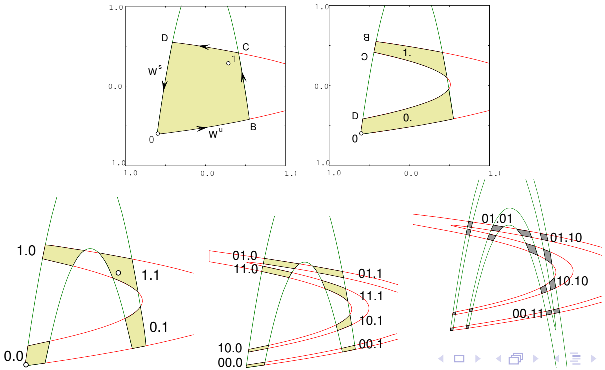
- We may then *choose* k so as to write the previous as the Hénon map

$$\begin{aligned} x'_{n+1} &= 1 - \alpha\, x'^2_n + y'_n \\ y'_{n+1} &= \beta\, x'_n \end{aligned} \tag{14}$$

The Hamiltonian Hénon map

- for $\alpha = 6$, $\beta = -1$, determine the non-wandering set:

$$\Omega = \left\{ x | \, x \in \lim_{m,n\to\infty} f^m(\mathcal{M}) \cap f^{-n}(\mathcal{M}) \right\} \qquad (15)$$



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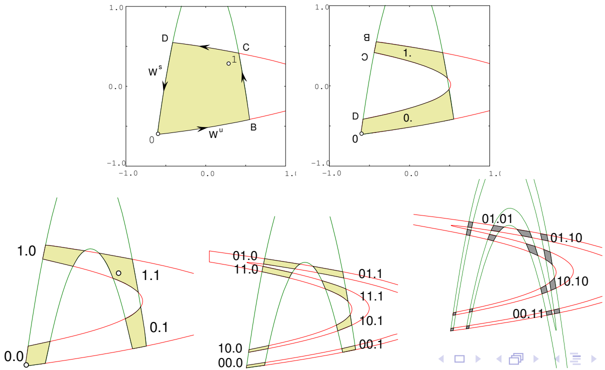
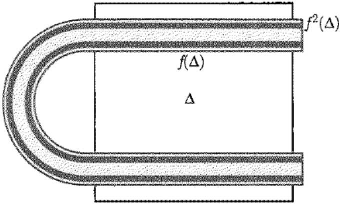
- can draw successive $\Omega_{m,n}$, intersections of horseshoes



Smale's horseshoes

- Topologically, the dynamics produces a sequence of horseshoes



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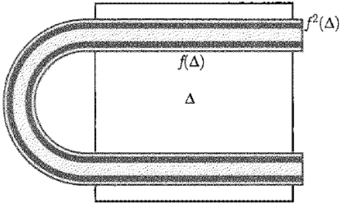

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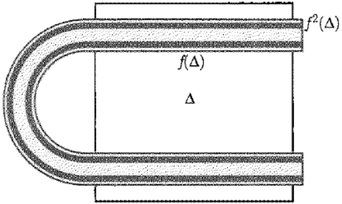

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- Likewise for the vertical $\bigcap_{i=0}^{n} f^{-i}(\Delta)$, so that $\Omega = \bigcap_{i=-\infty}^{\infty} f^i(\Delta)$ is a Cantor set, and h is a homeomorphism:

$$h : S \longrightarrow \Omega\,, \quad h(\{s\}) = \bigcap_{n\in\mathbb{Z}} f^n(\Delta_{s_n}) \tag{16}$$

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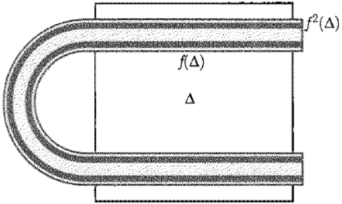

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- *Corollary*: periodic points of f are dense in Ω, and $f_{|\Omega}$ is topologically mixing

The baker's transformation

- Definition

$$
\begin{aligned}
x_{n+1} &= 2x_n - \theta\left(x_n - \frac{1}{2}\right) \\
y_{n+1} &= \frac{1}{2}y_n + \frac{1}{2}\theta\left(x_n - \frac{1}{2}\right)
\end{aligned}
\qquad (17)
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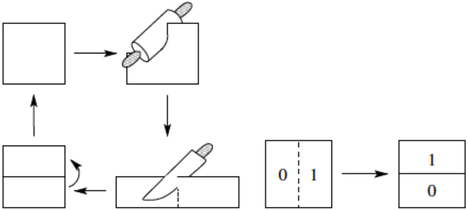
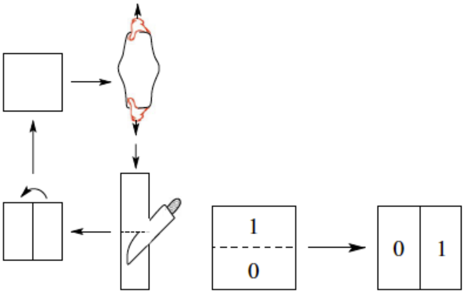


Rössler & Letellier 2020

The baker's transformation

- inverse transformation: swap x and y

$$\begin{aligned} x_{n+1} &= \frac{1}{2}x_n + \frac{1}{2}\theta\left(y_n - \frac{1}{2}\right) \\ y_{n+1} &= 2y_n - \theta\left(y_n - \frac{1}{2}\right) \end{aligned} \qquad (18)$$



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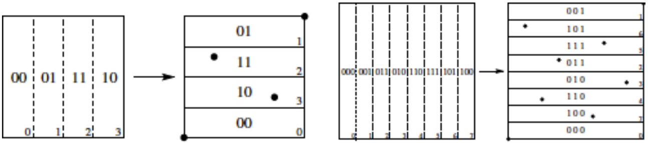
Kneading Danish pastry

- Rolling out, cutting, and stacking up, there are 2^n stripes



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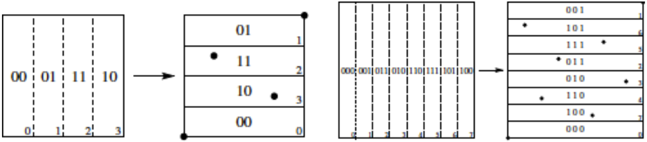
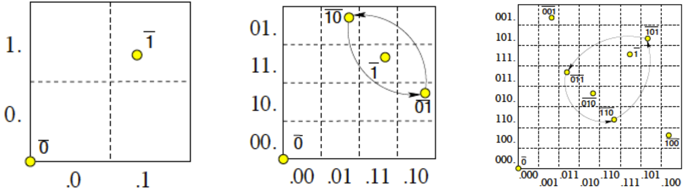


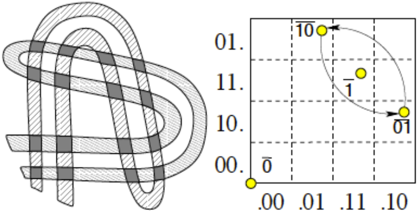
- go back *and* forth in time, build symbol square of rectangles $[s_{-m+1} \cdots s_0.s_1 s_2 \cdots s_n]$, each of size $2^{-m} \times 2^{-n}$



Rössler & Letellier, ChaosBook.org

Pruning

- the Kneading operation comes from intersections between manifolds



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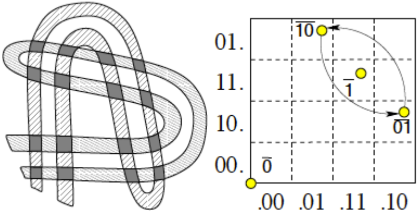
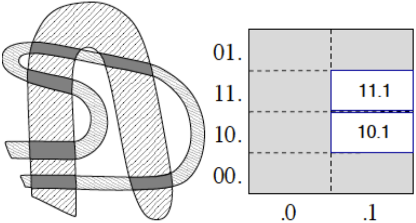


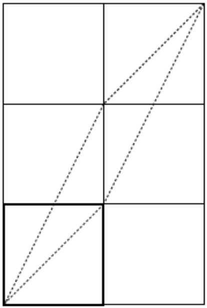
- but some intersections may miss out

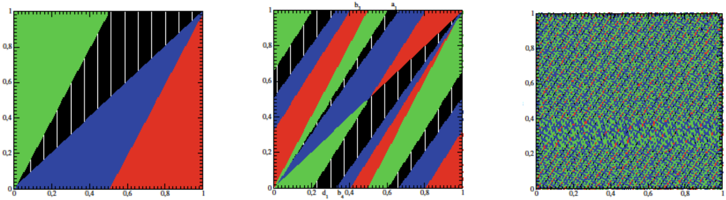


Continuous Automorphism on a Torus

- Definition

$$
\begin{aligned}
x_{n+1} &= x_n + y_n \mod 1 \\
y_{n+1} &= x_n + 2y_n \mod 1
\end{aligned}
$$

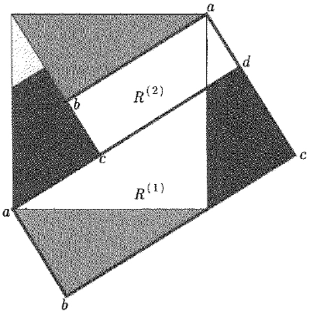




Coding the CAT

Build a partition:

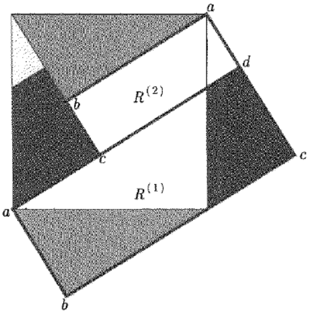
- draw stable/unstable manifolds



Coding the CAT

Build a partition:

- draw stable/unstable manifolds
- map the areas from intersections back into the torus



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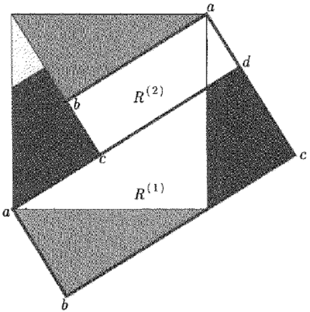

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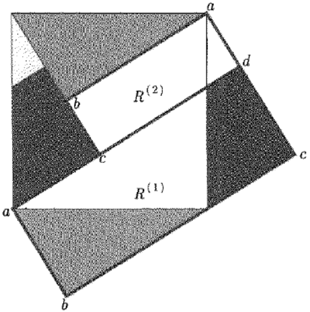

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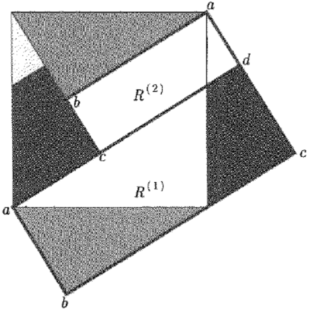
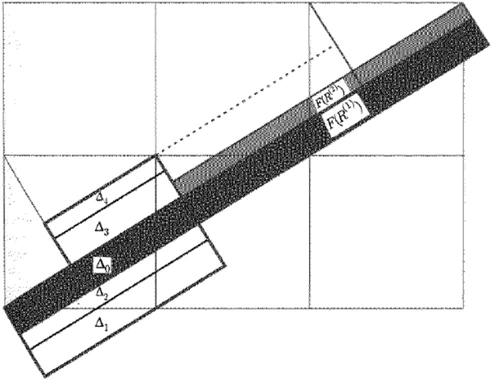


- We have two rectangles $R^{(1)}, R^{(2)}$
- To make a partition, look at $F\left(R^{(i)}\right)$

Coding the CAT



- $F\left(R^{(1)}\right)$ consists of three components, two in $R^{(1)}$ and one in $R^{(2)}$

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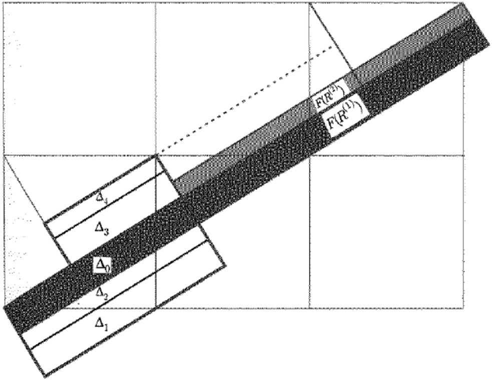

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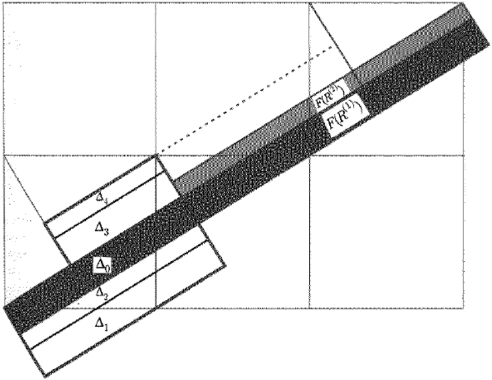

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- $F\left(R^{(2)}\right)$ consists of two components, one in $R^{(1)}$ and one in $R^{(2)}$
- Totally five components: $\Delta_0, \Delta_1, \Delta_2, \Delta_3, \Delta_4$

Coding the CAT

- Every intersection $\bigcap_{n \in \mathbb{Z}} f^n(\Delta_{s_n})$ contains no more than one point

Coding the CAT

- Every intersection $\bigcap_{n\in\mathbb{Z}} f^n(\Delta_{s_n})$ contains no more than one point
- one can define

$$h : S_A \longrightarrow \mathbb{T}^2 \tag{19}$$

such that

$$f \circ h = h \circ \sigma_A \tag{20}$$

- where

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \tag{21}$$