a spatiotemporal theory of
turbulence
ChaosBook.org/chaos1

Predrag Cvitanović

last lecture of the course
Georgia Tech

April 26, 2022

overview

1. what this course is about
2. turbulence in large domains

how do clouds solve PDEs?

do clouds integrate Navier-Stokes equations?

?

 $\Longrightarrow$ other swirls $\Longrightarrow$ 

are clouds Navier-Stokes supercomputers in the sky?

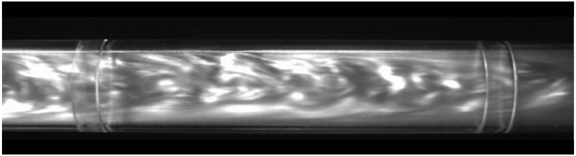
1. turbulence in large domains
2. spacetime

goal : enumerate the building blocks of turbulence

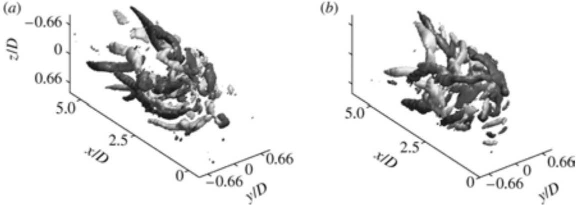
describe turbulence

starting from the equations (no statistical assumptions)

challenge : experiments are amazing

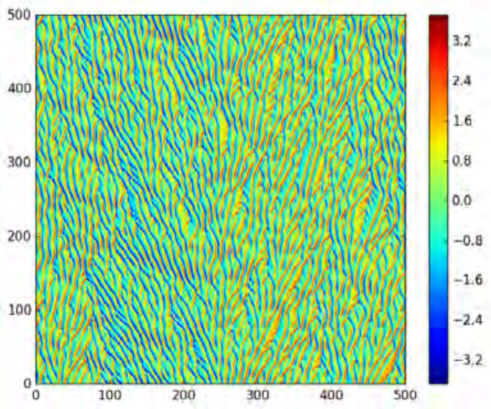


T. Mullin lab



B. Hof lab

an example : Kuramoto-Sivashinsky on a large domain

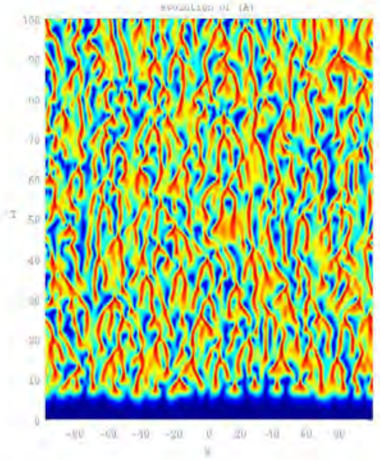


[horizontal] space $x \in [0, L]$ [up] time evolution

Gudorf 2018

another example of large spacetime domain turbulence

complex Ginzburg-Landau



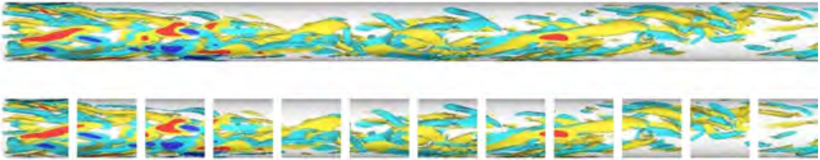
[horizontal] space $x \in [-L/2, L/2]$ [up] time evolution

codeinthehole.com/static/tutorial/coherent.html

1. turbulence in large domains
2. **spacetime**

fluid dynamics in large turbulent domains

pipe flow close to onset of turbulence [1]



we have a detailed theory of small turbulent fluid cells

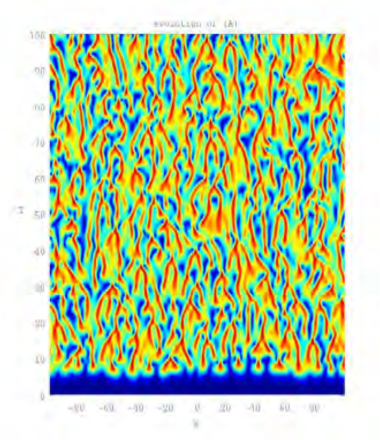
can we can we construct the infinite pipe by coupling small turbulent cells ?

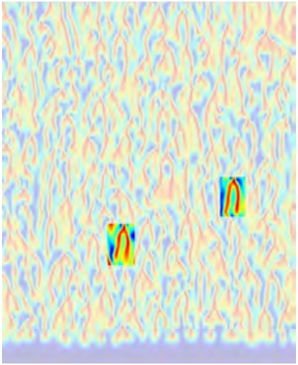
what would that theory look like ?

[1] M. Avila and B. Hof, Phys. Rev. **E 87** (2013)

complex Ginzburg-Landau on a large spacetime domain

goal : enumerate nearly recurrent patterns

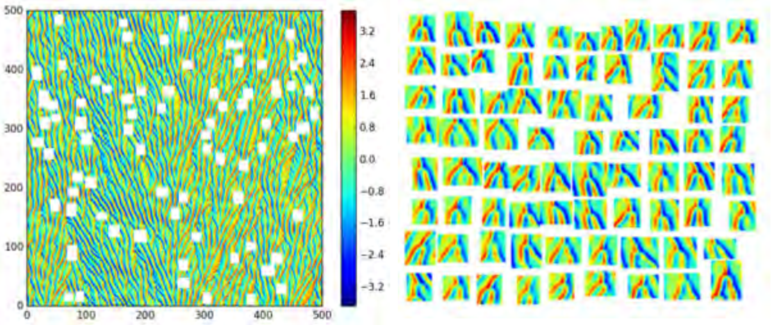




[left-right] space $x \in [-L/2, L/2]$ [up] time $t \in [0, T]$

Kuramoto-Sivashinsky on a large spacetime domain

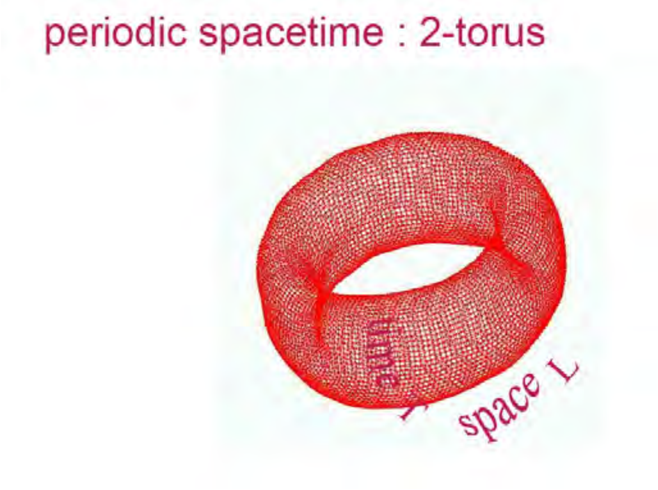
the same small tile recurs often in a turbulent pattern



goal : define, enumerate nearly recurrent tiles

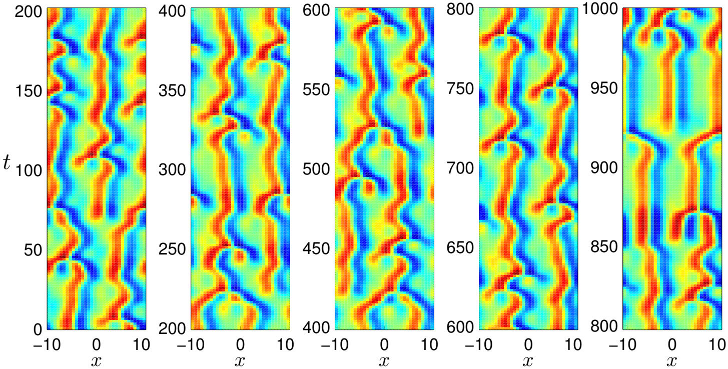
Gudorf 2018

use spatiotemporally compact solutions as lego blocks



this ‘exact coherent structure’
shadows a small patch of spacetime solution $u(x, t)$

evolution of Kuramoto-Sivashinsky on small $L = 22$ cell



horizontal: $x \in [-11, 11]$
vertical: time
color: magnitude of $u(x, t)$

periodic orbits generalize to d-tori

1 time, 0 space dimensions

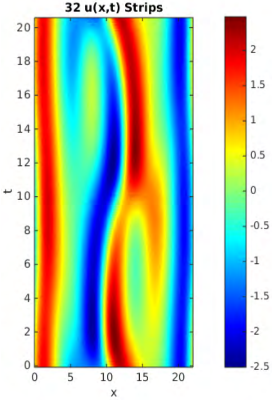
a state space point is *periodic* if its orbit returns to it after a finite time T ;
such orbit tiles the time axis by infinitely many repeats

1 time, d-1 space dimensions

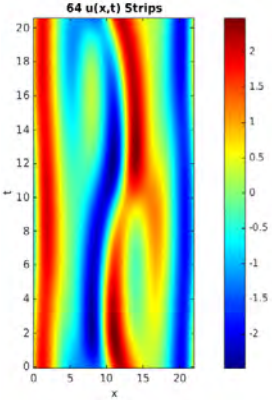
a state space point is *spatiotemporally periodic* if it belongs to an invariant d-torus $\mathcal{R}$;
such torus tiles the spacetime by infinitely many repeats

a spacetime invariant 2-torus integrated in either time or space

(left)



(right)



(left) old : time evolution $t = [0, T]$

initial condition : space periodic line $x = [0, L]$

(right) new : space evolution $x = [0, L]$

initial condition : time periodic line $t = [0, T]$

Gudorf 2016

1. turbulence in large domains
2. spacetime
3. **spacetime computations**

how do clouds solve PDEs?

clouds do not **NOT** integrate Navier-Stokes equations

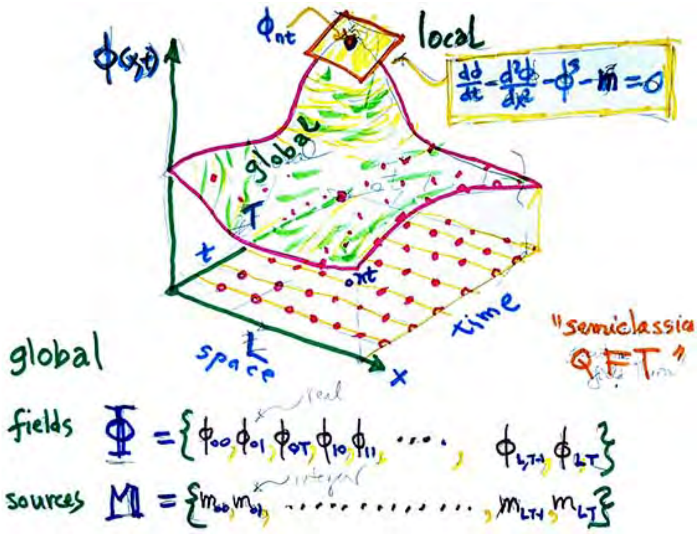
$\Longrightarrow$ other swirls $\Longrightarrow$

do clouds satisfy Navier-Stokes equations?

yes!

they satisfy them **locally**, everywhere and at all times

think globally, act locally



for each symbol array M, a periodic lattice state X_M

the equations are imposed as local constraints

your equation here, Feynman form:

$$F(u) = 0$$

for example, minimize over the entire 2-torus

cost function

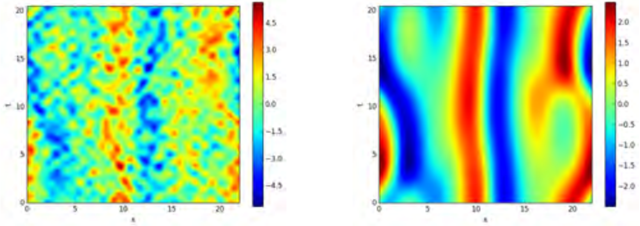
$$G \equiv \frac{1}{2} |F(u)|^2_{L \times T}$$

does it work at all ?

add strong noise to a *known* solution,
twice the typical amplitude

test 1

(not how we actually generate guesses)



(left) initial guess: a known invariant 2-torus

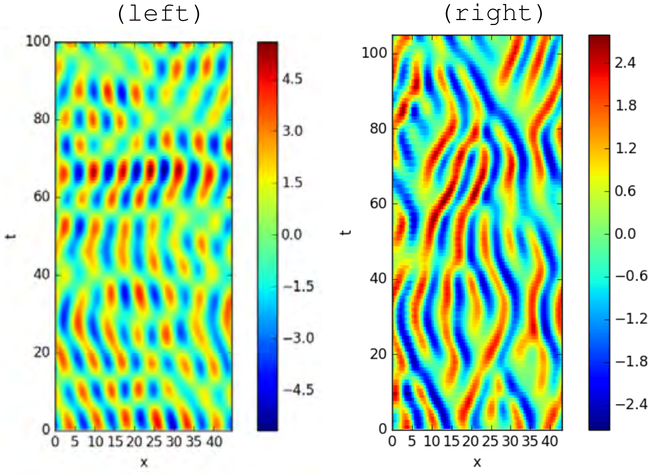
$(L_0, T_0) = (22.0, 20.5057459345)$ + strong random noise

(right) the resulting adjoint descent converged invariant 2-torus

$(L_f, T_f) = (21.95034935834641, 20.47026321555662)$

Gudorf 2018

test 2 - invariant 2-torus found variationally



(left) initial : $\tilde{L} = 2\pi\sqrt{2}$ spatially modulated “noisy” guess
(right) adjoint descent : converged invariant 2-torus

Gudorf 2018

1. turbulence in large domains
2. spacetime
3. fundamental tiles

building blocks of turbulence

how do we recognize a cloud?

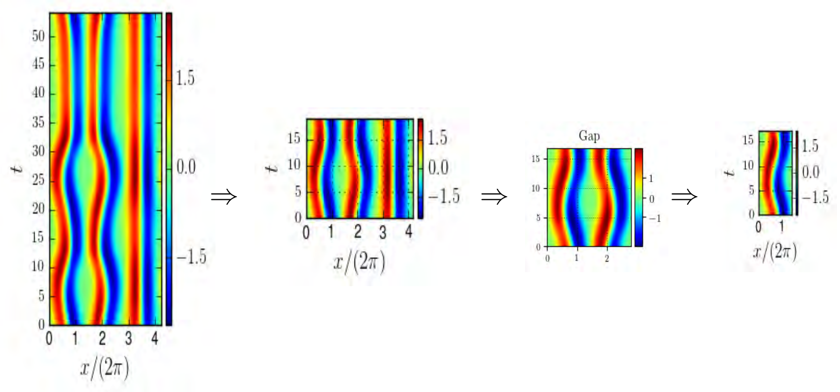
WATCH

$\Longrightarrow$ other swirls $\Longrightarrow$

by recurrent shapes!

so, construct an alphabet of possible shapes

extracting a fundamental tile

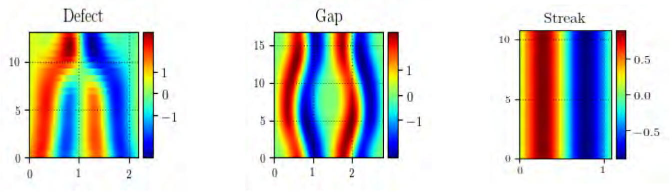


1) invariant 2-torus
2) invariant 2-torus computed from initial guess cut out from 1)
3) “gap" invariant 2-torus, initally cut out from 2)
4) the “gap" prime invariant 2-torus fundamental domain

Gudorf 2018

a trial set of 'prime' tiles

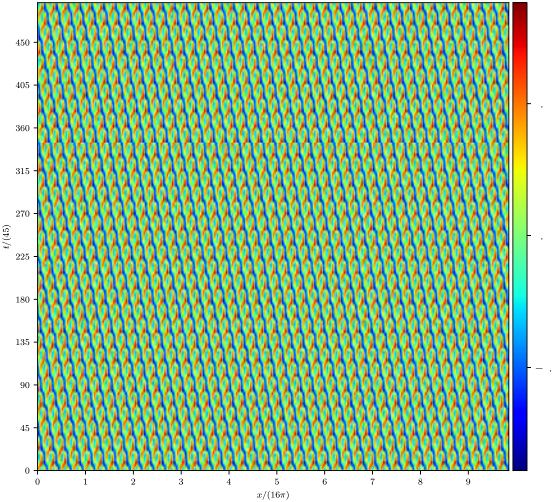
an alphabet of Kuramoto-Sivashinsky fundamental tiles



utilize also discrete symmetries :
spatial reflection, spatiotemporal shift-reflect, $\cdots$

Gudorf 2018

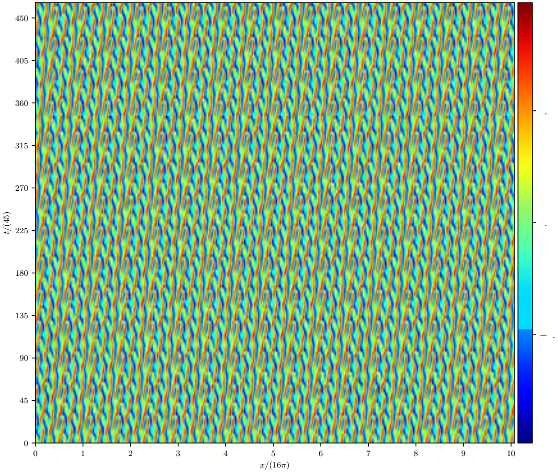
Kuramoto-Sivashinsky tiled by a small tile



tiling by relative periodic invariant 2-torus
$(L, T) = (13.02, 15)$

Gudorf 2018

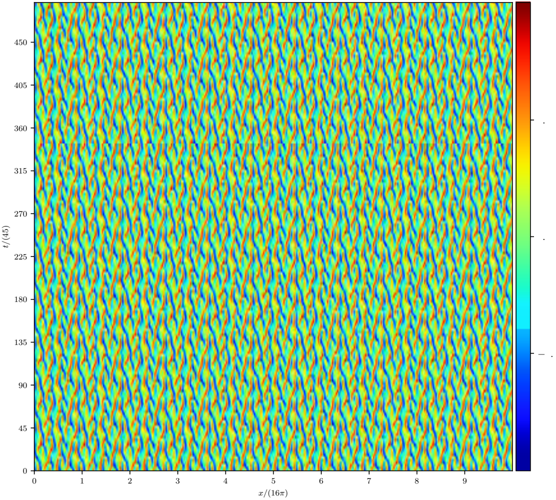
spacetime tiled by a larger tile



tiling by relative periodic invariant 2-torus
$(L, T) = (33.73, 35)$

Gudorf 2018

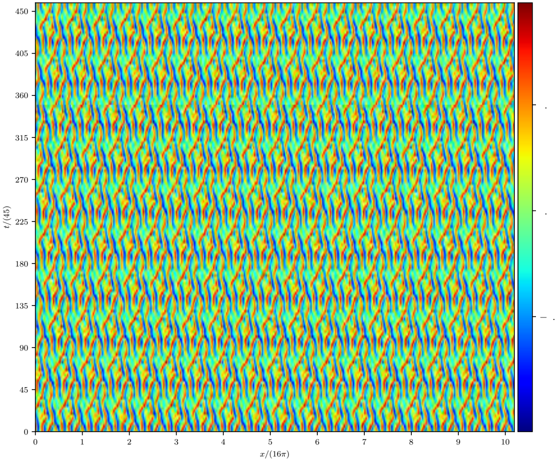
spacetime tiled by a tall tile



tiling by shift-reflect invariant 2-torus
$(L, T) = (55.83, 24)$

Gudorf 2018

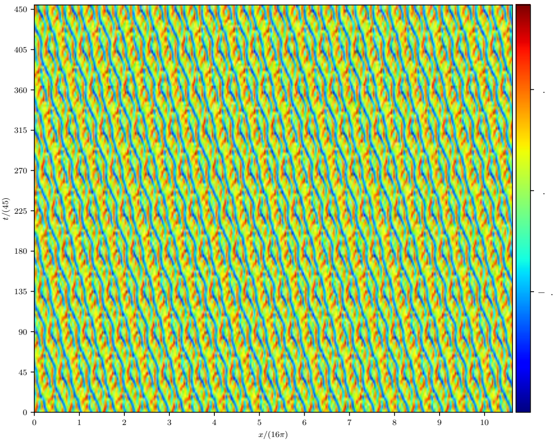
spacetime tiled by a larger tile



tiling by relative periodic invariant 2-torus
$(L, T) = (32.02, 51)$

Gudorf 2018

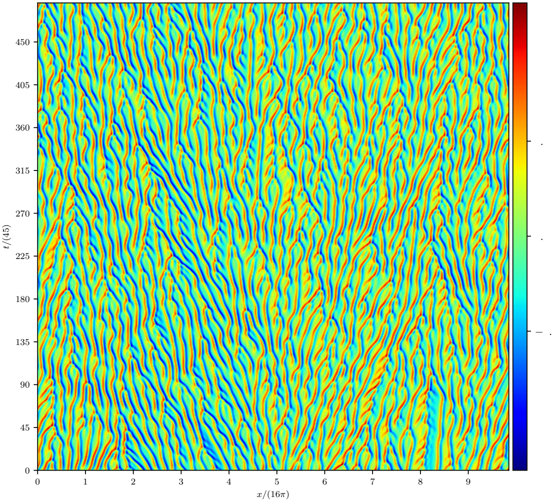
spacetime tiled by a larger tile



tiling by relative periodic invariant 2-torus
$(L, T) = (44.48, 50)$

Gudorf 2018

any single tiling looks nothing like turbulent Kuramoto-Sivashinsky !



[horizontal] space $x \in [-L/2, L/2]$ [up] time evolution

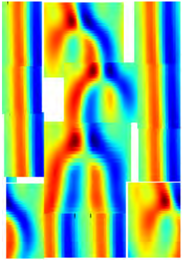
Gudorf 2018

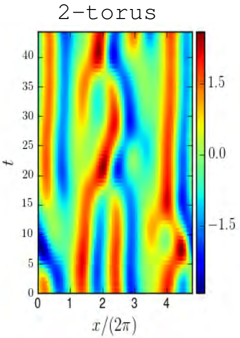
1. turbulence in large domains
2. spacetime
3. fundamental tiles
4. **gluing tiles**

Gudorf 2018

a qualitative tiling guess

a tiling and the resulting solution



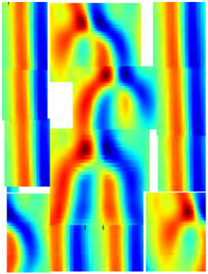


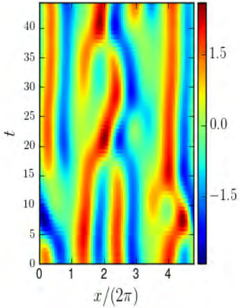
Gudorf 2018

enumerate hierarchically spatiotemporal patterns

2D symbolic encoding $\Rightarrow$ admissible solutions

<table>
<tr><td>0</td><td colspan="2">2</td><td>0</td></tr>
<tr><td>0</td><td colspan="2">1</td><td>0</td></tr>
<tr><td>0</td><td colspan="2" rowspan="2">1*</td><td>0</td></tr>
<tr><td>0</td><td>0</td></tr>
<tr><td rowspan="2">1</td><td>0</td><td>0</td><td rowspan="2">1</td></tr>
<tr><td>0</td><td>0</td></tr>
</table>





- each symbol indicates a minimal spatiotemporal tile
- glue them in all admissible ways

take home : clouds do not integrate PDEs

do clouds integrate Navier-Stokes equations?

NO!

$\Longrightarrow$ other swirls $\Longrightarrow$

at any spacetime point Navier-Stokes equations describe the local tangent space

they satisfy them locally, everywhere and at all times

course part 1 geometry of chaos : summary

1. study turbulence in infinite spatiatemporal domains
2. theory : classify all spatiotemporal tilings
3. numerics : future is spatiotemporal

there is no more time

there is only enumeration of spacetime solutions

this solves all your problems :)

1. (semi-)classical field theories

Dreams of Grand Schemes : solve

Navier-Stokes

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = \rho X_i - \frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i$$

Einstein

$$R_{ik} - \frac{1}{2} g_{ik} R = \frac{8\pi k}{c^4} T_{ik}$$

$$R^i_{\ klm} = \frac{\partial \Gamma^i_{km}}{\partial x^l} - \frac{\partial \Gamma^i_{kl}}{\partial x^m} + \Gamma^i_{nl} \Gamma^n_{km} - \Gamma^i_{nm} \Gamma^n_{kl}$$

Yang-Mills

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F_a^{\mu\nu}$$

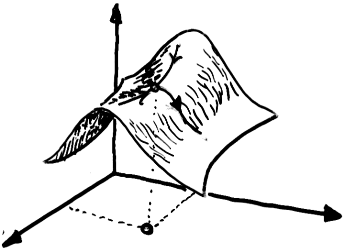
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, c_{abc} A^b_\mu A^c_\nu$$

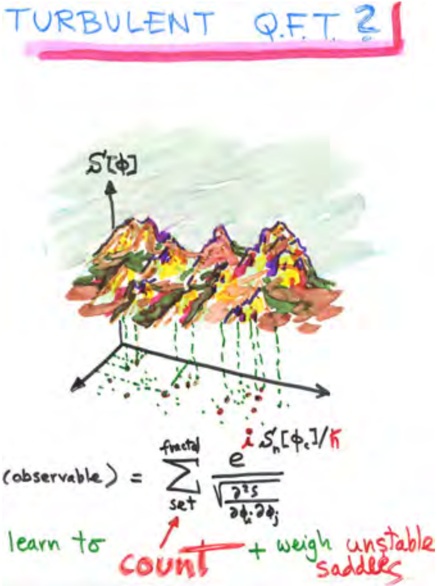
QUANTUM ?

QFT path integrals : semi-classical quantization

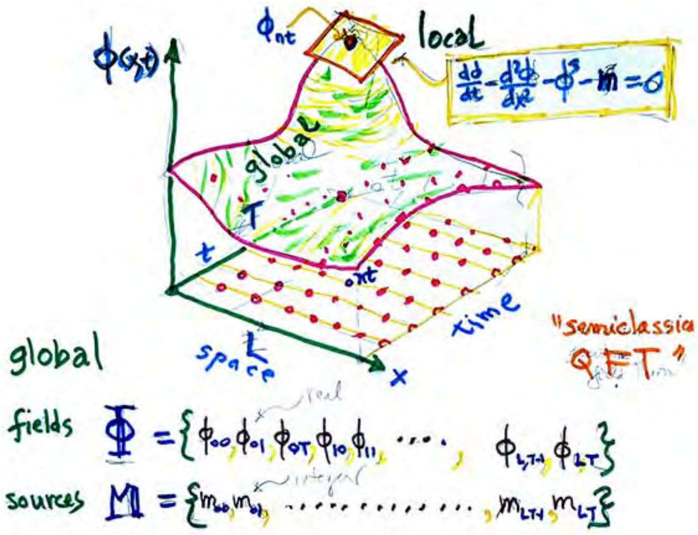
a fractal set of saddles

a local unstable extremum





think globally, act locally



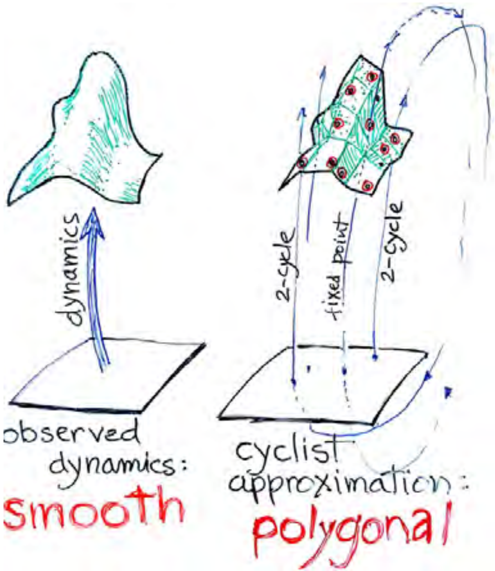
for each symbol array M, a periodic lattice state X_M

a theory of turbulence ?

the very short answer : POT

if you win : I teach you how

(for details, see ChaosBook.org/course1/index2.html)



tessellate the state space by

spatiotemporal periodic orbits

classical trace formula for continuous time flows

$$\sum_{\alpha=0}^{\infty} \frac{1}{s - s_\alpha} = \sum_p T_p \sum_{r=1}^{\infty} \frac{e^{r(\beta A_p - s T_p)}}{\left|\det\left(\mathbf{1} - M_p^r\right)\right|}$$

relates the spectrum of the evolution operator

$$\mathcal{L}(x', x) = \delta\big(x' - f^t(x)\big)\ e^{\beta A(x,t)}$$

to the unstable periodic orbits p of the flow $f^t(x)$.

classical trace formula for averaging over 2-tori

something like

$$\sum_{\alpha=0}^{\infty} \frac{1}{s - s_\alpha} = \sum_p V_p \sum_{r=1}^{\infty} \frac{e^{r(\beta A_p - s V_p)}}{|\det \mathcal{J}_{p^r}|}$$

weighs the unstable relative prime (all symmetries quotiented) d-torus p by the inverse of its Hill determinant, the determinant (state space volume) of its orbit Jacobian matrix $\mathcal{J}_p$

$$\det \mathcal{J}_p$$

and V_p is the volume

$$V_p = T_p L_p$$

of the prime spacetime tile p